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Bi-differential calculi, bi-Hamiltonian systems and conformal Killing tensors

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Abstract. The theory of Dimakis and Müller-Hoissen (Dimakis A and Müller-Hoissen F 2000 *J. Phys. A: Math. Gen.* **33** 957–74) concerning bi-differential calculi and completely integrable systems is related to bi-Hamiltonian systems of the Poisson–Nijenhuis type. In the special case where the ambient manifold is a cotangent bundle one is able to recover and elucidate the theory of Ibort *et al* (Ibort A, Magri F and Marmo G 2000 *J. Geom. Phys.* **33** 210–23), which is in turn a reworking in the bi-Hamitonian context of Benenti's theory of Hamilton–Jacobi separable systems. In particular, it is shown that Benenti's conformal Killing tensor, which is central to his theory, has an even more special form than has hitherto been realized and that when it is converted into a field of endomorphisms by raising an index with the ambient metric, it necessarily has vanishing Nijenhuis torsion.

1. Introduction

In a recent paper in this journal Dimakis and Müller-Hoissen [5] have shown how to generate conservation laws in completely integrable systems by using a bi-differential calculus. In [3] we described briefly how this approach to integrable systems is related to the standard approach using bi-Hamiltonian structures of the Poisson–Nijenhuis type, for systems with finitely many degrees of freedom. Here we shall extend that work as follows: we shall discuss the Poisson–Nijenhuis case, corresponding to a simple bi-differential calculus, in greater detail; and we shall describe a certain two-dimensional version of what Dimakis and Müller-Hoissen call a gauged bi-differential calculus. Our constructions basically involve a symplectic manifold (M, ω) and a type $(1, 1)$ tensor field R on M. When M is a cotangent bundle T^*Q and R is the complete lift of a tensor field J on Q, the gauged bi-differential calculus we choose turns out to be equivalent to the construction of bi-Hamiltonian systems on $T^*Q \times \mathbb{R}$, recently discussed by Ibort *et al* [7] and related by these authors to a theorem of Gelfand and Zakharevich. Ibort *et al* use the Gelfand–Zakharevitch construction to explain some results of Benenti [1,2] on the separation of variables in the Hamilton–Jacobi equation for Hamiltonians of mechanical type, when the metric defining the kinetic energy admits a conformal Killing tensor with certain properties. By considering the relationship between the work of Ibort *et al* and that of Dimakis and Müller-Hoissen we have been able to sharpen some of the results of the former.

One of our main conclusions is that the conformal Killing tensor that lies at the heart of Benenti's construction is not merely of gradient type but has an even more special form. In particular, such a conformal Killing tensor necessarily has vanishing Nijenhuis torsion when it is converted into a field of endomorphisms via the ambient metric. We shall also prove a partial converse property, namely, that a conformal Killing tensor with vanishing Nijenhuis torsion in the above sense and whose eigenvalues are functionally independent is necessarily of the special type.

2. Simple bi-differential calculi and Poisson–Nijenhuis structures

Let M be a smooth manifold; it will be convenient for large parts of this paper to assume that M is simply connected, so that closed 1-forms are exact. We consider the exterior algebra $\wedge(M)$ of forms on M, over the algebra $C^\infty(M)$ of real-valued C^∞ functions on M. By a simple (as opposed to gauged) bi-differential calculus on $\wedge(M)$, we mean (following [5]) a pair (d_1, d_2) of derivations of $\wedge(M)$ of degree 1, which both have the co-boundary property ${d_i}^2 = 0$ and which commute in the graded sense, by which we mean that $[d_1, d_2] := d_1d_2 + d_2d_1 = 0$. The fundamental observation of [5], for our purposes, is that if $\chi^{(0)} \in C^\infty(M)$ satisfies $d_1d_2\chi^{(0)} = -d_2d_1\chi^{(0)} = 0$, then one can inductively define a sequence of functions $\chi^{(m)}$, $m = 0, 1, 2, \ldots$, according to the rule

$$d_1\chi^{(m+1)} = d_2\chi^{(m)}.$$

In particular, let d_1 be the ordinary exterior derivative d. We know from Frölicher–Nijenhuis theory [6] that every other derivation of degree 1 which commutes with d (i.e. is a derivation of type d_*) must be of the form d_R for some type (1, 1) tensor field R on M. Furthermore, the necessary and sufficient condition for d_R to satisy ${d_R}^2 = 0$ is that the torsion, or Nijenhuis tensor, N_R of R must be zero. We recall that the action of d_R on $C^\infty(M)$ is given by $d_Rf = R^*(df)$, where we think of the tensor R as a homomorphism of the $C^\infty(M)$-module of vector fields on M, and R^* as its adjoint acting on 1-forms.

Now let $\chi^{(0)}$ be a function satisfying $dd_R\chi^{(0)} = -d_Rd\chi^{(0)} = 0$, and consider the sequence of functions $\chi^{(m)}$ defined according to the rule stated above, which now reads

$$d\chi^{(m+1)} = d_R\chi^{(m)}.$$

This sequence has interesting properties in the particular case in which M is a Poisson manifold, as we now explain.

Proposition 2.1. *Suppose that $(\wedge(M), d, d_R)$ is a bi-differential calculus on a simply connected manifold M; and that Ω is a bi-vector field on M such that $\Omega(R^*(\alpha), \beta) = \Omega(\alpha, R^*(\beta))$ for any 1-forms α, β. Let $\chi^{(0)}$ satisfy $dd_R\chi^{(0)} = 0$. Then the functions $\chi^{(m)}$ defined by $d\chi^{(m+1)} = d_R\chi^{(m)}$ satisfy*

$$\Omega(d\chi^{(m)}, d\chi^{(n)}) = 0 \qquad \textit{for all} \quad m, n \geqslant 0.$$

Proof. Note first that the assumption on Ω implies that for any functions ϕ, ψ,

$$\Omega(d_R\phi, d\psi) = \Omega(d\phi, d_R\psi).$$

Clearly we have $\Omega(d\chi^{(0)}, d\chi^{(0)}) = 0$. Suppose that the assertion is true for $m, n \leqslant k$; then for $n < k$

$$\begin{aligned}\Omega(d\chi^{(k+1)}, d\chi^{(n)}) &= \Omega(d_R\chi^{(k)}, d\chi^{(n)}) \\ &= \Omega(d\chi^{(k)}, d_R\chi^{(n)}) = \Omega(d\chi^{(k)}, d\chi^{(n+1)}) = 0.\end{aligned}$$

In exactly the same way we get $\Omega(d\chi^{(k+1)}, d\chi^{(k)}) = \Omega(d\chi^{(k)}, d\chi^{(k+1)})$, which must therefore also be zero in view of the skew symmetry of Ω, and of course $\Omega(d\chi^{(k+1)}, d\chi^{(k+1)}) = 0$. It follows that the assertion is also true for $m, n \leqslant k + 1$. □

Corollary. Suppose in addition that $[\Omega, \Omega] = 0$, where $[\cdot, \cdot]$ is the Schouten bracket, so that if $\{\phi, \psi\} = \Omega(d\phi, d\psi)$ then $\{\cdot, \cdot\}$ is a Poisson bracket. Then the functions $\chi^{(m)}$ are in involution. In particular, if (M, ω) is symplectic, and R is symmetric with respect to ω, then the functions $\chi^{(m)}$ are in involution with respect to the Poisson bracket defined by ω.

We have shown in [3] that a bi-differential calculus $(\wedge(M), d, d_R)$ endows a symplectic manifold (M, ω) such that $\omega(R(\cdot), \cdot) = \omega(\cdot, R(\cdot)) = \omega_1$ and $d\omega_1 = 0$ with a Poisson–Nijenhuis structure. That is to say, there is a second Poisson bracket, which is compatible with the given one, in the sense that any linear combination of the two, with constant coefficients, is also a Poisson bracket. The recursion tensor of the structure is the R we started from, and the inductively defined functions $\chi^{(m)}$ are in involution with respect to both Poisson brackets. We now wish to discuss in more detail the interplay between the various assumptions underlying these statements.

We first recall the following result concerning Poisson structures from [9] (see also [8,10]). Let P denote the $C^\infty(M)$-linear map of 1-forms to vector fields defined by a Poisson structure (i.e. $\Omega(\alpha, \beta) = \langle P(\alpha), \beta\rangle$), and suppose that R is a type $(1, 1)$ tensor field such that $PR^* = RP$. For any 1-form α and vector field X on M, we define a vector field $\mu_{R,P}(\alpha, X)$ by

$$\mu_{R,P}(\alpha, X) = (\mathcal{L}_{P(\alpha)}R)(X) - P(\mathcal{L}_X(R^*\alpha)) + P(\mathcal{L}_{R(X)}\alpha).$$

As a consequence of the assumption that $PR^* = RP$, $\mu_{R,P}$ is a type $(1, 2)$ tensor field on M, sometimes called the Magri–Morosi concomitant (see [12]) of R and P. Then in order for RP to define a Poisson structure it is sufficient that R satisfies the following two conditions:

(1) the torsion of R is zero;
(2) the Magri–Morosi concomitant of R and P is zero.

We now consider more specifically the case in which P comes from a symplectic form ω on M. We shall establish some interesting equivalent ways of expressing the second condition, that $\mu_{R,P} = 0$; it is important, in the context of this paper, to note that these results do not require that the Nijenhuis torsion of R vanishes.

Since $\mu_{R,P}$ is a tensor, it is sufficient to consider its value when its 1-form argument is exact. For any function f on M, $P(df) = X_f$, the Hamiltonian vector field associated with f by P. Since P here comes from a symplectic structure, $\mathrm{i}_{P(\alpha)}\omega = -\alpha$. Furthermore, $\mu_{R,P} = 0$ if and only if $\mathrm{i}_{\mu_{R,P}(df,Y)}\omega = 0$ for all f and Y. Now

$$\begin{aligned}\mathrm{i}_{\mu_{R,P}(df,Y)}\omega &= \mathrm{i}_{\mathcal{L}_{X_f}R(Y)}\omega + \mathcal{L}_Y(d_Rf) - \mathcal{L}_{R(Y)}df \\ &= \mathrm{i}_{\mathcal{L}_{X_f}R(Y)}\omega + \mathrm{i}_Y dd_Rf + d(\mathrm{i}_Y d_Rf) - d(\mathrm{i}_{R(Y)}df) \\ &= \mathrm{i}_{\mathcal{L}_{X_f}R(Y)}\omega + \mathrm{i}_Y dd_Rf.\end{aligned}$$

Note that the symmetry of R with respect to ω and the invariance of ω under the flow of X_f together imply that $\mathcal{L}_{X_f}R$ is also symmetric with respect to ω. Thus $\mu_{R,P} = 0$ if and only if

$$\mathrm{i}_{\mathcal{L}_{X_f}R}\omega = -2dd_Rf$$

for all functions f; this is our first equivalent representation of the vanishing of $\mu_{R,P}$.

The defining relation for ω_1 likewise reads $\omega_1 = \frac{1}{2}\mathrm{i}_R\omega$, from which it follows, using the definition $d_R = [\mathrm{i}_R, d]$, that $d\omega_1 = -\frac{1}{2}d_R\omega$. Making use of the commutator identity (see [6]) $[\mathrm{i}_X, d_R] := \mathrm{i}_X d_R + d_R\mathrm{i}_X = -\mathrm{i}_{\mathcal{L}_XR} + \mathcal{L}_{R(X)}$, we next obtain

$$\begin{aligned}\mathrm{i}_{X_f}d\omega_1 &= -\tfrac{1}{2}\mathrm{i}_{X_f}d_R\omega = -\tfrac{1}{2}d_Rdf + \tfrac{1}{2}\mathrm{i}_{\mathcal{L}_{X_f}R}\omega - \tfrac{1}{2}d\mathrm{i}_{R(X_f)}\omega \\ &= \tfrac{1}{2}dd_Rf + \tfrac{1}{2}\mathrm{i}_{\mathcal{L}_{X_f}R}\omega - \tfrac{1}{2}\mathcal{L}_{X_f}\omega_1 + \tfrac{1}{2}\mathrm{i}_{X_f}d\omega_1\end{aligned}$$

from which it follows that

$$\mathrm{i}_{X_f} d\omega_1 = dd_R f + \tfrac{1}{2}\mathrm{i}_{\mathcal{L}_{X_f} R}\omega.$$

Hence, the vanishing of the right-hand side is further equivalent to $d\omega_1 = 0$. In conclusion, we have proved the following result.

Proposition 2.2. *Let (M, ω) be a symplectic manifold with corresponding Poisson map P. We assume that R is a type $(1, 1)$ tensor field such that $PR^* = RP$. Then the following conditions are equivalent:*

(1) the Magri–Morosi concomitant $\mu_{R,P}$ vanishes;
(2) $\mathrm{i}_{\mathcal{L}_{X_f} R}\omega = -2dd_R f$ for all f;
(3) $d\omega_1 = 0$, where $\omega_1 = \frac{1}{2}\mathrm{i}_R\omega$.

If in addition to the above equivalent conditions it is assumed that $N_R = 0$, then RP defines a second Poisson structure which is compatible with the original one. The second Poisson bracket on M is given by

$$\{f, g\}_1 = RP(df)g = R(X_f)g = \omega_1(X_f, X_g) = -\langle X_g, d_R f\rangle.$$

We then have a bi-Hamiltonian manifold of Poisson–Nijenhuis type. (For clarity, we should point out that if R is assumed to be non-singular, then ω_1 is symplectic and therefore also defines a second Poisson structure, which this time is $R^{-1}P$. This Poisson structure need not be compatible with P, however. We shall be concerned only with the compatible structure whose bracket $\{\cdot, \cdot\}_1$ is made explicit above.)

We consider a bi-differential calculus (d, d_R) on a symplectic manifold (M, ω), where R satisfies any, and hence all, of the conditions of proposition 2.2. Then there is a corresponding Poisson–Nijenhuis structure. Suppose that R has n functionally independent real eigenfunctions, each of which has geometric multiplicity two. As we showed in [3], one then can generate through the iterative procedure associated with the bi-differential calculus the sums of the powers of the eigenfunctions of R, and they are in involution with respect to both Poisson brackets. It follows that the eigenfunctions themselves are in involution.

As an example of the Poisson–Nijenhuis structure just described, we consider the case in which M is a cotangent bundle, $M = T^*Q$, with its standard symplectic structure $\omega = d\theta$. Let J be a type $(1, 1)$ tensor field on Q, and write $\hat{J}$ for the linear transformation of fibres of T^*Q induced by J. We consider the 2-form ω_1 on T^*Q defined by

$$\omega_1 = d(\hat{J}^*\theta)$$

and define a tensor R by $\omega_1 = \omega(R(\cdot), \cdot)$. It was observed via a coordinate calculation in [7] that this tensor is the complete lift of J to T^*Q, which we will denote by $\tilde{J}$. In fact it can be shown by intrinsic methods, starting from the intrinsic definition of the complete lift $\tilde{J}$ given, for example, in [4], that the 2-form ω_1 introduced above is the same as the $\omega_1 = \frac{1}{2}\mathrm{i}_{\tilde{J}}\omega$ occurring in proposition 2.2. We discuss these issues in an appendix to this paper. For the moment, however, we record that we have constructed a tensor $R = \tilde{J}$ on the symplectic manifold $(T^*Q, d\theta)$ which is manifestly symmetric with respect to $\omega = d\theta$, and that moreover the corresponding $d\omega_1$ is zero. We thus know from proposition 2.2 that the Magri–Morosi concomitant will vanish. Furthermore, accepting that $\tilde{J}$ is indeed the complete lift of J, we know that $N_{\tilde{J}} = 0$ if and only if $N_J = 0$ (see, e.g., [4] or [14]). Thus any type $(1, 1)$ tensor field on Q with zero torsion defines a Poisson–Nijenhuis structure on T^*Q.

If J has n functionally independent real eigenfunctions then $\tilde{J}$ has the same eigenfunctions, each of which is doubly degenerate. In fact when $N_J = 0$ and J has n functionally independent real eigenfunctions, we can take the eigenfunctions as coordinates q^i on Q, and with respect to these coordinates

$$J = \sum_{i=1}^{n} q^i \frac{\partial}{\partial q^i} \otimes dq^i.$$

Then in terms of the corresponding canonical coordinates (q^i, p_i) on T^*Q,

$$\tilde{J} = \sum_{i=1}^{n} q^i \left(\frac{\partial}{\partial q^i} \otimes dq^i + \frac{\partial}{\partial p_i} \otimes dp_i \right)$$

and $\omega_1 = \sum_{i=1}^{n} q^i dq^i \wedge dp_i$. (The fact that the eigenfunctions are in involution with respect to the standard Poisson bracket is not very interesting here—they are so because they are independent of p_i.)

This set-up is in fact locally typical, in the following sense. We return to the case of an arbitrary symplectic manifold M and a type (1, 1) tensor field R which satisfies all the assumptions which lead to the conclusion that its eigenfunctions are in involution. If these eigenfunctions are functionally independent then by Liouville's theorem we can use them as one-half of a set of canonical coordinates, so that $\omega = \sum_{i=1}^{n} dq^i \wedge dp_i$ with q^i the eigenfunctions of R. Then using the fact that $N_R = 0$ it can be shown that the p_i can be chosen so that R and ω_1 are given as above for $\tilde{J}$. Such coordinates are called Darboux–Nijenhuis coordinates (see [11]).

The equation $dd_R f = 0$ plays an important role in the generation of the sequence of functions in involution using the bi-differential calculus method (in the simple case, and also in fact in the gauged case we will discuss below). It is therefore worth noting that our results tell us the general solution of this equation, considered as an equation for f. For given R with $N_R = 0$, satisfying the other conditions assumed above, such functions f, in Darboux–Nijenhuis coordinates, are of the form

$$f = \sum_{i=1}^{n} f_i(q_i)$$

where the q_i are the eigenfunctions of R. As a byproduct, we see from the equivalences established in proposition 2.2, that these functions are precisely the Hamiltonian functions f such that $\mathcal{L}_{X_f} R = 0$.

3. Kinetic energy Hamiltonians

In the previous section we considered the condition $dd_R f = 0$ as an equation for f, given R. In fact the other interpretation, namely that in which f is given and R is the unknown, is more interesting. If we also require that R should satisfy the equivalent conditions of proposition 2.2, then we are looking for tensors R which are invariant under X_f. Note that proposition 2.2 does not require the vanishing of the torsion of R; in analysing this situation we can therefore start without assuming that the Nijenhuis condition is in effect.

We consider the particular case in which $f(q, p) = h(q, p) = \frac{1}{2} g(p, p)$ is a Hamiltonian on T^*Q of kinetic energy type, and $R = \tilde{J}$ is the complete lift of a type (1, 1) tensor field on Q. We seek those J for which $dd_{\tilde{J}} h = 0$. The simplest way of carrying out the calculations is to use tensor methods. Thus we write

$$h = \tfrac{1}{2} g^{ij} p_i p_j$$

with the summation convention now in force; we regard the g^{ij} as the components of the covariant form of the metric tensor g on Q, and use this metric freely to raise and lower indices.

(In fact it is the need to raise and lower indices, or in other words to pass conveniently between the contravariant and covariant versions of tensors relative to the metric, which makes the use of the tensor calculus efficient here.) Furthermore we shall use the Levi–Civita connection associated with the metric, denoting the connection coefficients by Γ^i_{jk} and the covariant derivative by a vertical bar.

We work in terms of 1-forms on T^*Q adapted to the connection, which are given by dq^i and π_i where

$$\pi_i = dp_i - \Gamma^k_{ij} p_k dq^j$$

the dual basis of vector fields is $\{X_i, \partial/\partial p_i\}$ where

$$X_i = \frac{\partial}{\partial q^i} + \Gamma^k_{ij} p_k \frac{\partial}{\partial p_j}.$$

Then if $J = J^i_j \partial/\partial q^i \otimes dq^j$,

$$\tilde{J} = J^i_j \left(X_i \otimes dq^j + \frac{\partial}{\partial p_j} \otimes \pi_i \right) + (J^k_{i|j} - J^k_{j|i}) p_k \frac{\partial}{\partial p_i} \otimes dq^j.$$

We have $dh = g^{ij} p_j \pi_i$, and it follows that

$$d_{\tilde{J}} h = J^{ij} p_j \pi_i + g_{il} (J^{kj|l} - J^{kl|j}) p_j p_k dq^i.$$

Now

$$d\pi_i = \tfrac{1}{2} R^l_{ijk} p_l dq^j \wedge dq^k + \Gamma^k_{ij} dq^j \wedge \pi_k$$

where R^l_{ijk} are the components of the curvature tensor. It follows that

$$\begin{aligned} dd_{\tilde{J}} h = {} & -J^{ij} \pi_i \wedge \pi_j + g_{il} (-J^{jk|l} + J^{jl|k} + J^{kl|j}) p_k dq^i \wedge \pi_j \\ & + (\tfrac{1}{2} J^{kl} R^m_{kij} + g_{jk} (J^{lm|k} - J^{lk|m})_{|i}) p_l p_m dq^i \wedge dq^j. \end{aligned}$$

Thus in order that $dd_{\tilde{J}} h = 0$, J must satisfy $J^{ji} = J^{ij}$ and

$$-J^{jk|l} + J^{jl|k} + J^{kl|j} = 0.$$

The first two terms in the latter equation taken together are skew symmetric in k and l, while the third term is symmetric. It follows that $J^{kl|j} = 0$, that is, J must be a parallel tensor field with respect to the connection. The further condition that $J^{kl} R^m_{kij} + J^{km} R^l_{kij} = 0$ is automatically satisfied when the first two are also satisfied: in fact it is the integrability condition for a symmetric tensor to be parallel. Summarizing, we have proved the following result.

Proposition 3.1. *Let g be a given metric tensor field on Q and $h = \frac{1}{2} g^{ij} p_i p_j$ the corresponding kinetic energy Hamiltonian on T^*Q. Then, for a type $(1, 1)$ tensor field J on Q to have the property $dd_{\tilde{J}} h = 0$, it is necessary and sufficient that J is symmetric and parallel.*

This result, though not very exciting in itself perhaps, has one very interesting feature: the torsion of $\tilde{J}$ vanishes automatically, as a consequence of the conditions for $dd_{\tilde{J}} h$ to vanish. In the penultimate section of the paper we shall discuss the case of a Hamiltonian of mechanical type, which is rather more complicated, and leads to involutory integrals of genuine interest; in that case also it will turn out, remarkably, that the vanishing of the torsion of the recursion tensor is a consequence of an analoguous condition on $dd_{\tilde{J}} h$.

4. A gauged bi-differential calculus and bi-Hamiltonian structures on an extended space

In a gauged bi-differential calculus (see [5]), the pair of derivations (d_1, d_2) of degree 1 is replaced by operators $D_i = d_i + A_i$, where the A_i in general are $N \times N$ matrices of 1-forms and the operators D_i act on N-component column vectors of forms. (Actually in [5] the D_i act on square matrices of forms, but as we pointed out in [3] the construction works just as well as described above.) The D_i further have to satisfy the conditions

$$D_i{}^2 = 0 \qquad [D_1, D_2] := D_1 D_2 + D_2 D_1 = 0$$

(which are formally the same as the conditions satisfied by the derivations in a simple bi-differential calculus).

We consider the following scheme, with $N = 2$. We take $d_1 = d$ and $d_2 = d_R$, as before, and assume that (d, d_R) is a bi-differential calculus already (i.e. that $d_R{}^2 = 0$). We set $D_1 = d$, but (for arbitrary k-forms α, β) set

$$D_2 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = d_R \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} df & 0 \\ dh & 0 \end{bmatrix} \wedge \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} d_R\alpha + df \wedge \alpha \\ d_R\beta + dh \wedge \alpha \end{bmatrix}$$

for some fixed functions f and h. The conditions $D_1{}^2 = 0$ and $[D_1, D_2] = 0$ are automatically satisfied; we have $D_2{}^2 = 0$ if and only if

$$dd_R h = dh \wedge df \qquad dd_R f = 0.$$

Starting with a suitable vector of functions, we will be able to generate a sequence of such vectors by a procedure similar to the one in section 2. We want to show that the functions so obtained are in some sense in involution, so that we are dealing again with complete integrability. We shall first show that, assuming that we start from the situation where we already have a Poisson–Nijenhuis structure on M as described in section 2, the conditions on f and h derived above are necessary and sufficient to enable us to define a certain type of bi-Hamiltonian structure on $M \times \mathbb{R}$.

Proposition 4.1. *Let (M, ω) be a symplectic manifold and R a type $(1, 1)$ tensor with vanishing torsion, symmetric with respect to ω, and such that $\omega_1 = \frac{1}{2}\mathrm{i}_R\omega$ is closed. We assume the functions f and h satisfy $dd_R h = dh \wedge df$ and $dd_R f = 0$. Then $M \times \mathbb{R}$ can be equipped with a pair of compatible Poisson brackets.*

Proof. Let $\{\cdot, \cdot\}$ be the Poisson bracket on M associated with ω and $\{\cdot, \cdot\}_1$ the second one, as described in section 2. We will extend these Poisson brackets to $M \times \mathbb{R}$. To do so, we must decribe how they act on the extra coordinate function on $\mathbb{R}$, which we denote by z. First, we set $\{\cdot, z\} = 0$, that is we specify that z should be a Casimir of $\{\cdot, \cdot\}$. Then clearly the extended bracket is still Poisson (i.e. it satisfies the Jacobi identity). Now we define the function $\hat{h}$ on $M \times \mathbb{R}$ by $\hat{h} = h + zf$. We set

$$\{\cdot, z\}_1 = \{\cdot, \hat{h}\}$$

and show that the given conditions on f and h ensure that this defines a Poisson bracket on $M \times \mathbb{R}$. It is sufficient to consider whether the Jacobi identity holds with arguments z, ϕ and ψ, where ϕ and ψ are independent of z. Now $\{\phi, z\}_1 = \{\phi, h\} + z\{\phi, f\}$, since z is a Casimir of $\{\cdot, \cdot\}$. Thus

$$\begin{aligned} \{\psi, \{\phi, z\}_1\}_1 &= \{\psi, \{\phi, h\}\}_1 + \{\psi, z\{\phi, f\}\}_1 \\ &= \{\psi, \{\phi, h\}\}_1 + \{\psi, h\}\{\phi, f\} + z\{\psi, \{\phi, f\}\}_1 + z\{\phi, f\}\{\psi, f\}. \end{aligned}$$

It follows that

$$\{\psi,\{\phi,z\}_1\}_1+\{\phi,\{z,\psi\}_1\}_1+\{z,\{\psi,\phi\}_1\}_1$$
$$=\{h,\{\psi,\phi\}_1\}+\{\psi,\{\phi,h\}\}_1-\{\phi,\{\psi,h\}\}_1+\{\psi,h\}\{\phi,f\}-\{\phi,h\}\{\psi,f\}$$
$$+z(\{f,\{\psi,\phi\}_1\}+\{\psi,\{\phi,f\}\}_1-\{\phi,\{\psi,f\}\}_1).$$

In order for the Jacobi identity to hold, the terms independent of z must vanish and the coefficient of z must also vanish. The Poisson brackets on the right-hand side all involve functions independent of z, so can be evaluated using the formulae of section 2. With X_ϕ denoting as before the Hamiltonian vector field of ϕ calculated with respect to the Poisson bracket coming from the symplectic form ω, we have

$$\{\psi,\{\phi,h\}\}_1-\{\phi,\{\psi,h\}\}_1+\{h,\{\psi,\phi\}_1\}$$
$$=\omega_1(X_\psi,[X_\phi,X_h])-\omega_1(X_\phi,[X_\psi,X_h])+X_h(\omega_1(X_\psi,X_\phi))$$
$$=(\mathcal{L}_{X_h}\omega_1)(X_\psi,X_\phi)=(d\mathrm{i}_{X_h}\omega_1)(X_\psi,X_\phi)=-dd_Rh(X_\psi,X_\phi)$$

and

$$-\{\psi,h\}\{\phi,f\}+\{\phi,h\}\{\psi,f\}=df\wedge dh(X_\psi,X_\phi).$$

Thus the conditions for the existence of the gauged bi-differential calculus defined earlier are precisely those required to ensure that $\{\cdot,\cdot\}_1$ satisfies the Jacobi identity.

Finally, we must show that the extended Poisson brackets are compatible, that is, that for any constants λ and μ, $\lambda\{\cdot,\cdot\}_1+\mu\{\cdot,\cdot\}$ is also a Poisson bracket. Now the restriction of this bracket to M is the Poisson bracket corresponding to the recursion operator $\lambda R+\mu I$, where I is the identity tensor. On the other hand,

$$\lambda\{\cdot,z\}_1+\mu\{\cdot,z\}=\lambda\{\cdot,z\}_1=\{\cdot,\lambda\hat{h}\}.$$

Thus $\lambda\{\cdot,\cdot\}_1+\mu\{\cdot,\cdot\}$ is constructed according to the procedure described above, with the recursion operator $\lambda R+\mu I$ and the functions λh and λf. But these quantities satisfy the conditions just derived for the new bracket to be Poisson, if R, h and f do. So the extended Poisson brackets are compatible. □

Remark. It is easy to see that the Poisson tensor associated to the bracket $\{\cdot,\cdot\}_1$ on $M\times\mathbb{R}$ is given by

$$\Omega_1=\Omega(R^*(\cdot),\cdot)+\frac{\partial}{\partial z}\wedge(X_h+zX_f).$$

As an alternative proof of the above proposition, one can verify that the Schouten bracket $[\Omega_1,\Omega_1]$ is zero and that $[\Omega,\Omega_1]=0$ also.

We now turn to the properties of a vector sequence of functions $[f^{(m)},h^{(m)}]^T$ constructed recursively using the gauged bi-differential calculus, with a suitable choice of initial functions, and under the assumption that the manifold is simply connected. The idea is to define $[f^{(m+1)},h^{(m+1)}]^T$ by the rule $D_1[f^{(m+1)},h^{(m+1)}]^T=D_2[f^{(m)},h^{(m)}]^T$, or

$$d\begin{bmatrix}f^{(m+1)}\\h^{(m+1)}\end{bmatrix}=d_R\begin{bmatrix}f^{(m)}\\h^{(m)}\end{bmatrix}+\begin{bmatrix}df&0\\dh&0\end{bmatrix}\begin{bmatrix}f^{(m)}\\h^{(m)}\end{bmatrix}$$

or equivalently

$$df^{(m+1)}=d_Rf^{(m)}+f^{(m)}df\qquad dh^{(m+1)}=d_Rh^{(m)}+f^{(m)}dh$$

which can be started provided that $D_1D_2[f^{(0)},h^{(0)}]^T=0$. Take, for example, $f^{(0)}=1$ and $h^{(0)}=0$. Then $f^{(1)}=f$ and $h^{(1)}=h$. (Notice in passing the following differences between the recursive construction here and the one we discussed for a simple bi-differential calculus in section 2, and more explicitly in [3]: the condition $dd_R f = 0$ was the requirement on f to start the recursive procedure in section 2, whereas now it is part of the conditions for having a gauged bi-differential calculus, and we here initialize the recursion by choosing trivial values for $f^{(0)}$ and $h^{(0)}$.)

Proposition 4.2. *We consider the functions $f^{(m)}$ and $h^{(m)}$ as defined above and put*

$$\hat{h}^{(m)} = h^{(m)} + zf^{(m)}.$$

We then have the following properties:

(1) both the $h^{(m)}$ and $f^{(m)}$ are in involution with respect to both Poisson brackets on M;
(2) $\{h^{(i)}, f^{(j)}\} + \{f^{(i)}, h^{(j)}\} = 0$ for every $i, j \geqslant 1$, and the same property holds with respect to the second bracket on M;
(3) the functions $\hat{h}^{(m)}$ on $M \times \mathbb{R}$ are in involution with respect to both brackets.

Proof. We write $\chi^{(m)}$ to stand for either $h^{(m)}$ or $f^{(m)}$. The rule for generating $\chi^{(m+1)}$, when expressed in terms of Poisson brackets, is

$$\{\chi^{(m+1)}, \cdot\} = \{\chi^{(m)}, \cdot\}_1 + f^{(m)}\{\chi^{(1)}, \cdot\}.$$

We assume that $\{\chi^{(i)}, \chi^{(j)}\} = \{\chi^{(i)}, \chi^{(j)}\}_1 = 0$ for all i, j with $1 \leqslant i, j \leqslant m$; we show that the same is true with $m+1$ in place of m. First, for $1 \leqslant i \leqslant m$

$$\{\chi^{(m+1)}, \chi^{(i)}\} = \{\chi^{(m)}, \chi^{(i)}\}_1 + f^{(m)}\{\chi^{(1)}, \chi^{(i)}\} = 0.$$

Then

$$0 = \{\chi^{(i+1)}, \chi^{(m+1)}\} = \{\chi^{(i)}, \chi^{(m+1)}\}_1 + f^{(i)}\{\chi^{(1)}, \chi^{(m+1)}\}$$

whence $\{\chi^{(i)}, \chi^{(m+1)}\}_1 = 0$.

Second, let $k(i, j)$ stand for the function $\{h^{(i)}, f^{(j)}\} + \{f^{(i)}, h^{(j)}\}$. Then

$$\begin{aligned} k(i+1, j) &= \{h^{(i)}, f^{(j)}\}_1 + \{f^{(i)}, h^{(j)}\}_1 + f^{(i)}(\{h^{(1)}, f^{(j)}\} + \{f^{(1)}, h^{(j)}\}) \\ &= k(i, j+1) + f^{(i)}k(1, j) + f^{(j)}k(i, 1). \end{aligned}$$

Now suppose that $k(i, j) = 0$ for all i, j with $1 \leqslant i, j \leqslant m$. It follows from the formula above that $k(m+1, j) = 0$ for $1 \leqslant j < m$, while

$$k(m+1, m) = \{h^{(m)}, f^{(m)}\}_1 + \{f^{(m)}, h^{(m)}\}_1 + f^{(m)}k(1, m) = 0$$

and of course $k(m+1, m+1) = 0$. Finally, the first line in the expression for $k(i+1, j)$ shows that the vanishing of the $k(i, j)$ implies that also $\{h^{(i)}, f^{(j)}\}_1 + \{f^{(i)}, h^{(j)}\}_1 = 0$.

For the third part, we observe first that for any function ϕ on M

$$\begin{aligned} \{\phi, \hat{h}^{(m)}\}_1 &= \{\phi, h^{(m)}\}_1 + \{\phi, z\}_1 f^{(m)} + z\{\phi, f^{(m)}\}_1 \\ &= \{\phi, h^{(m+1)}\} - f^{(m)}\{\phi, h^{(1)}\} + z(\{\phi, f^{(m+1)}\} - f^{(m)}\{\phi, f^{(1)}\}) \\ &\quad + f^{(m)}(\{\phi, h^{(1)}\} + z\{\phi, f^{(1)}\}) \\ &= \{\phi, \hat{h}^{(m+1)}\} \end{aligned}$$

while

$$\{z, \hat{h}^{(m)}\}_1 = \{h^{(1)}, h^{(m)}\} + z(\{h^{(1)}, f^{(m)}\} + \{f^{(1)}, h^{(m)}\}) + z^2\{f^{(1)}, f^{(m)}\} = 0$$

as a result of the preceding two properties, so that trivially also $\{z, \hat{h}^{(m)}\}_1 = \{z, \hat{h}^{(m+1)}\}$. It follows that the $\hat{h}^{(m)}$ satisfy the recursion relation

$$\{\cdot, \hat{h}^{(m+1)}\} = \{\cdot, \hat{h}^{(m)}\}_1.$$

The fact that these functions are in involution can now be deduced from this relation by an inductive argument, or proved directly using the first two properties. □

Corollary. Suppose that the recursive generation of new functions breaks down at order $m+1$, by which we mean that $\hat{h}^{(m+1)} = 0$. Then

$$C(\lambda) = \sum_{i=0}^{m} \lambda^i \hat{h}^{(i)}$$

is a Casimir of the Poisson pencil $\Omega - \lambda\Omega_1$.

This result is to some extent related to a theorem of Gelfand and Zakharevich about the existence of a polynomial Casimir on an odd- dimensional bi-Hamiltonian manifold with a Poisson pencil of maximal rank (see [7,11]). The main result in [7] is based on this theorem and gives an interesting geometrical interpretation of the theory of Benenti about the separability of the Hamilton–Jacobi equation (see [2] and references therein). We will highlight some additional features of the results obtained by Ibort *et al* in the next section. However, it should be emphasized that the scheme we have described in this section already covers the main structural properties which are needed for that purpose and this scheme is valid for an arbitrary symplectic manifold M and with respect to a general type (1, 1) tensor field R.

5. Conformal Killing tensors with vanishing torsion

By way of generalization of the results of section 3, we now look at the conditions for the existence of the gauged bi-differential calculus, again from the perspective that h is given and R, and in this case f, are unknown.

We consider the case in which h is a Hamiltonian of mechanical type on T^*Q, and $R = \tilde{J}$. To keep the analogy with section 3, we will not assume from the outset that $N_J = 0$. As before, we assume that the kinetic energy part of h is determined by a metric, so that

$$h = \tfrac{1}{2} g^{ij} p_i p_j + V(q).$$

We will assume further that f is a function on Q. Then the condition $dd_{\tilde{J}}h = dh \wedge df$ becomes, using results obtained earlier:

$$\begin{aligned} -J^{ij}\pi^i \wedge \pi^j &+ g_{il}(-J^{jk|l} + J^{jl|k} + J^{kl|j}) p_k dq^i \wedge \pi_j \\ &+ (\tfrac{1}{2} J^{kl} R^m_{kij} + g_{jk}(J^{lm|k} - J^{lk|m})_{|i}) p_l p_m dq^i \wedge dq^j + dd_J V \\ &= -\frac{\partial f}{\partial q^i} dq^i \wedge (g^{jk} p_k \pi_j + dV). \end{aligned}$$

From the $\pi^i \wedge \pi^j$ terms we find that J must be symmetric, as before. The $dq^i \wedge \pi_j$ terms give

$$-J_{jk|l} + J_{jl|k} + J_{kl|j} = -g_{jk} \frac{\partial f}{\partial q^l}.$$

The symmetric part of this in k and l gives

$$J_{kl|j} = -\frac{1}{2}\left(g_{jk}\frac{\partial f}{\partial q^l} + g_{jl}\frac{\partial f}{\partial q^k}\right)$$

from which it follows that

$$\frac{\partial f}{\partial q^j} = -(J^k_k)_{|j} \qquad \text{or} \qquad f = -\operatorname{tr} J$$

(up to a constant). Furthermore,

$$J_{jk|l} + J_{jl|k} + J_{kl|j} = -\left(g_{jk}\frac{\partial f}{\partial q^l} + g_{kl}\frac{\partial f}{\partial q^j} + g_{lj}\frac{\partial f}{\partial q^k}\right)$$

which is to say that J is a conformal Killing tensor of gradient type. Moreover, if one substitutes for the covariant derivatives in the expression for the torsion of J, which is

$$J_i^l J_{j|l}^k - J_j^l J_{i|l}^k + J_l^k (J_{i|j}^l - J_{j|i}^l)$$

one finds that it vanishes. So once again the vanishing of the torsion of J is a consequence of the condition that $dd_{\bar{J}}h$ must satisfy. If one takes the trace of the torsion on j and k one finds that

$$d_J(\operatorname{tr} J) = \tfrac{1}{2}d(\operatorname{tr} J^2).$$

Thus the condition on f, which reduces to $dd_J f = 0$, is also satisfied automatically.

Finally, the terms in $dq^i \wedge dq^j$ involving J vanish as a result of the differential condition it satisfies, and we are left with the following condition on V:

$$dd_J V = dV \wedge df.$$

We have therefore proved the following result, which adds some interesting features to proposition 2 in [7].

Proposition 5.1. *Let g be a given metric tensor field on Q and V and f functions on Q, and let h be the Hamiltonian function on T^*Q given by $h = \frac{1}{2}g^{ij}p_i p_j + V(q)$. The necessary and sufficient conditions for a type $(1, 1)$ tensor field J on Q to have the property $dd_{\bar{J}}h = dh \wedge df$ are that J is symmetric and satisfies the equations*

$$J_{kl|j} = -\frac{1}{2}\left(g_{jk}\frac{\partial f}{\partial q^l} + g_{jl}\frac{\partial f}{\partial q^k}\right)$$

while the functions V and f satisfy $dd_J V = dV \wedge df$. Such a J is a conformal Killing tensor of g of gradient type and $-f$ is its trace (up to a constant). Moreover, J will have vanishing torsion and f has the property $dd_J f = 0$; this implies that all conditions are satisfied for the existence of a gauged bi-differential calculus of the type described in the previous section.

The special property of the conformal Killing tensors we encounter here is characteristic of all sufficiently general conformal Killing tensors with vanishing torsion, as we will now show.

Proposition 5.2. *Let J be a type $(1, 1)$ tensor field on an n-dimensional (pseudo)-Riemannian manifold (Q, g) such that the corresponding $(0, 2)$ tensor is conformal Killing with factor α. We assume further that J has n real, functionally independent eigenfunctions. Then $N_J = 0$ if and only if*

$$J_{kl|j} = \tfrac{1}{2}(\alpha_l g_{jk} + \alpha_k g_{jl}).$$

This further implies that J is conformal Killing of gradient type.

Proof. We know that J satisfies $\sum J_{jk|l} = \sum \alpha_l g_{jk}$, where the summation sign stands for the cyclic sum over all indices. Putting

$$T_{jkl} = J_{jk|l} - \tfrac{1}{2}(\alpha_j g_{kl} + \alpha_k g_{jl})$$

it follows that $\sum T_{jkl} = 0$. The Nijenhuis condition now can be written in the form $J_j^m(T_{mkl} - T_{mlk}) = J_l^m T_{jkm} - J_k^m T_{jlm}$. Adding to this the identity $J_j^m(T_{mkl} + T_{klm}) = -J_j^m T_{lmk}$, we obtain an expression for $2J_j^m T_{mkl}$ which is symmetric in j and k. It follows that

$$J_j^m T_{mkl} = J_k^m T_{mjl}.$$

This indicates that for each fixed l, the symmetric matrix T_l with components T_{jkl} commutes with J. Since J, by assumption, has distinct eigenvalues at each point of an open dense subset of Q, it follows that all T_l are simultaneously diagonalizable. It then easily follows from the symmetry properties of T_{jkl} that with respect to a basis of eigenvectors of J, all components of T_l are actually zero. Hence, J has the required property, from which it further follows that $\alpha_j = \partial f/\partial q^j$, with $f = \operatorname{tr} J$. The proof in the other direction is contained in earlier statements. □

We end this section by pointing out that the conditions on f and h in the case that R is the complete lift $\tilde{J}$ again have an interpretation in terms of the Lie derivative of $\tilde{J}$ with respect to the corresponding Hamiltonian vector fields.

Proposition 5.3. *The conditions $dd_{\tilde{J}} f = 0$ and $dd_{\tilde{J}} h = dh \wedge df$, for arbitrary functions f, h on T^*Q, are equivalent to*

$$\mathcal{L}_{X_f} \tilde{J} = 0 \qquad \mathcal{L}_{X_h} \tilde{J} = X_f \otimes dh - X_h \otimes df.$$

Proof. The statement about f has already been mentioned in section 3. Concerning X_h, we know from the second of the equivalent properties in proposition 2.2 that

$$\begin{aligned} \mathrm{i}_{\mathcal{L}_{X_h} \tilde{J}} d\theta &= -2dd_{\tilde{J}} h = -2dh \wedge df \\ &= dh \wedge \mathrm{i}_{X_f} d\theta - df \wedge \mathrm{i}_{X_h} d\theta \end{aligned}$$

from which the result easily follows. □

6. Applications

We first briefly review the results of [7] and [1], before mentioning the new insights that our work provides.

At the heart of the matter lie the recurrence relations

$$df^{(m+1)} = d_J f^{(m)} + f^{(m)} df \qquad dh^{(m+1)} = d_{\tilde{J}} h^{(m)} + f^{(m)} dh$$

where $h = \frac{1}{2} g^{ij} p_i p_j + V(q)$, J is a conformal Killing tensor of g with vanishing torsion and functionally independent eigenfunctions, $f = -\operatorname{tr} J$, and V satisfies $dd_J V = dV \wedge df$. Then, as Ibort *et al* show,

(1) for $m = 1, 2, \ldots, \dim M$ we can take for $f^{(m)}$ the mth elementary symmetric function of the eigenfunctions of J, and $f^{(m)} = 0$ for $m > \dim M$;

(2) with this choice of the $f^{(m)}$, $h^{(m)}$ takes the form $h^{(m)} = \frac{1}{2} K^{(m)ij} p_i p_j + V^{(m)}(q)$, $m = 1, 2, \ldots, \dim M$, where the $K^{(m)}$ are independent, pairwise commuting Killing tensors of g with common closed eigenforms, $K^{(1)} = g$, the $V^{(m)}$ satisfy $dV^{(m)} = d_{K^{(m)}} V$ so that $dd_{K^{(m)}} V = 0$, and $h^{(m)} = 0$ for $m > \dim M$.

Such a collection of Killing tensors is called a Stäckel system. It is known [2, 7] that if the metric g admits a Stäckel system and if the potential V satisfies $dd_{K^{(m)}} V = 0$ then the Hamilton–Jacobi equation for the Hamiltonian $h = \frac{1}{2} g^{ij} p_i p_j + V(q)$ is separable in orthogonal coordinates.

In fact in [1], Benenti proves the following result.

Let (u^i) be orthogonal coordinates on a Riemannian manifold (M, g). If

$$\frac{\partial}{\partial u^i}(\ln g^{jj}) = \frac{1}{u^j - u^i} \qquad i \neq j \qquad u^i \neq u^j$$

then the Hamilton–Jacobi equation for geodesics is separable.

Benenti shows that the tensor L whose components relative to the orthogonal coordinates are

$$L^{ii} = u^i g^{ii} \qquad L^{ij} = 0 \qquad i \neq j$$

is a conformal Killing tensor whose torsion vanishes, and is the generator of a Stäckel system.

This result is related to our work as follows. In proposition 5.2 we have proved that a conformal Killing tensor field L, which has n real, functionally independent eigenfunctions, satisfies $N_L = 0$ if and only if

$$L_{ij|k} = \tfrac{1}{2}(\alpha_i g_{jk} + \alpha_j g_{ik}).$$

Now the eigenfunctions of L may be used as coordinates (u^i); with respect to these coordinates L takes the form (as a type (1, 1) tensor)

$$L = \sum_{i=1}^{n} u^i \frac{\partial}{\partial u^i} \otimes du^i$$

and since L is symmetric (as a type (0, 2) tensor), the metric tensor g is diagonal. The 1-form α whose components appear in the formula for $L_{jk|l}$ is given by $\alpha = d(\operatorname{tr} L)$, so $\alpha = \sum du^i$. The formula for $L_{jk|l}$, re-expressed for convenience in the form

$$L^i_{j|k} = \tfrac{1}{2}(\alpha^i g_{jk} + \alpha_j \delta^i_k)$$

reduces to

$$(u^j - u^i)\Gamma^{\,i}_{jj} = \frac{g_{jj}}{2g_{ii}} \qquad (u^j - u^i)\Gamma^{\,i}_{ij} = -\tfrac{1}{2}$$

for $i \neq j$; when i, j and k are all different and when they are all the same the equations are identically satisfied. But when the coordinates are orthogonal,

$$\Gamma^{\,i}_{jj} = -\frac{1}{2g_{ii}}\frac{\partial g_{jj}}{\partial u^i} \qquad \Gamma^{\,i}_{ij} = \frac{1}{2g_{ii}}\frac{\partial g_{ii}}{\partial u^j}$$

for $i \neq j$, so the formula for $L_{jk|l}$ reduces to just the one condition

$$\frac{\partial}{\partial u^i}(\ln g_{jj}) = -\frac{1}{u^j - u^i}.$$

Since $g^{jj} = 1/g_{jj}$, this is identical to the condition in Benenti's result.

Ibort *et al* show that the recurrence relations $dh^{(m+1)} = d_{\bar{J}} h^{(m)} + f^{(m)} dh$ may be written in the form $\{\cdot, \hat{h}^{(m+1)}\} = \{\cdot, \hat{h}^{(m)}\}_1$ as in the proof of proposition 4.2; that is, that the $\hat{h}^{(m)}$ satisfy Lenard's recurrence relations for the bi-Hamiltonian structure on $M \times \mathbb{R}$. We have given an alternative way of obtaining these recurrence relations, namely by using a certain gauged bi-differential calculus. We have shown further that the consistency conditions for such a bi-differential calculus are identical with the consistency conditions for such a bi-Hamiltonian structure, in a rather more general context than that required for the separability argument. Finally, we have shown that the conformal Killing tensor that occurs in the construction of either the bi-differential calculus or the bi-Hamiltonian system must be of a special form, that as a consequence the vanishing of its torsion is automatic and not an additional requirement, and that sufficiently general conformal Killing tensors with vanishing torsion must take this special form.

The separability results require that the conformal Killing tensor has functionally independent eigenfunctions. By way of further application, we will illustrate how the results of sections 4 and 5 can be used in a constructive procedure that may lead to the identification of conservation laws even when this condition is not satisfied. The idea is to start with a given metric tensor field g, i.e. with the kinetic energy part T of a Hamiltonian on T^*Q, as the only data, and to proceed from there to construct suitable type (1, 1) tensor fields J, potentials V and functions $h^{(m)}$ which Poisson-commute with $h = T+V$. The procedure works as follows. First, the characteristic property of conformal Killing tensors with vanishing torsion, as identified in proposition 5.1, is used as a set of partial differential equations for the determination of suitable tensor fields J. The trace of such J defines corresponding functions f which then give rise to equations for the potential via the condition $dd_J V = dV \wedge df$. We finally appeal to the recursive scheme described in proposition 4.2 to construct functions $f^{(m)}$ and $h^{(m)}$.

Let us illustrate this procedure by taking g to be the Euclidean metric (in dimension n). Since raising or lowering indices in this case has no effect on the coordinate representation of tensor fields, we will write all indices for convenience as lower indices. The equations for J become (with $J_{kl} = J_{lk}$)

$$J_{kl,j} = -\frac{1}{2}\left(\delta_{jk}\frac{\partial f}{\partial q_l} + \delta_{jl}\frac{\partial f}{\partial q_k}\right)$$

for some function f. It readily follows then that we must have (indices with a different name in each equation are assumed to be different and there are no summations)

$$J_{kl,j} = 0 \qquad J_{ii,j} = 0 \qquad J_{ii,i} = -\frac{\partial f}{\partial q_i}(q_i) \qquad J_{kl,k} = -\frac{1}{2}\frac{\partial f}{\partial q_l}(q_l).$$

These equations are easy to solve and have the following general solution:

$$J_{kl} = aq_kq_l + b_kq_l + b_lq_k + c_{kl}$$

where the a, b_k and $c_{kl} = c_{lk}$ are constants. Notice that for $a \neq 0$, by a Euclidean coordinate transformation, we can simplify this expression to

$$J_{kl} = aq_kq_l + c_k\delta_{kl}$$

so that J is what Benenti, in [1], calls a planar inertia tensor. The case in which the c_k (the eigenvalues of c_{kl}) are distinct leads, via a particular case of the results quoted above, to elliptic coordinates for Euclidean space.

Suppose, however, we proceed to the other extreme by taking $c_k = 0$ and $a = 1$. With $f = -\sum_i {q_i}^2$, the equations for V become

$$q_i\left(q_j\frac{\partial^2 V}{\partial q_i \partial q_k} - q_k\frac{\partial^2 V}{\partial q_i \partial q_j}\right) + 3\left(q_j\frac{\partial V}{\partial q_k} - q_j\frac{\partial V}{\partial q_k}\right) = 0$$

from which it follows that $q_j\partial V/\partial q_k - q_k\partial V/\partial q_j$ must be a homogeneous function of degree -2. This is so if V is of the form $V = V_1 + V_2$ where V_1 is an arbitrary function of $\sum_i {q_i}^2$ and V_2 is an arbitary function which is homogeneous of degree -2.

Next, we have to find functions $(f^{(m)}, h^{(m)})$ recursively from the equations

$$df^{(m+1)} = d_J f^{(m)} + f^{(m)}df \qquad dh^{(m+1)} = d_{\tilde{J}}h^{(m)} + f^{(m)}dh$$

with $f^{(1)} = f = -|\boldsymbol{q}|^2$ and $h^{(1)} = h = \frac{1}{2}|\boldsymbol{p}|^2 + V$ (we have introduced obvious vector notations here). The first hierarchy of functions immediately terminates, i.e. we find $f^{(2)} = 0$, whence $f^{(m)} = 0$ for $m \geqslant 2$. (Note that in this case J has a single non-zero eigenfunction, namely $|\boldsymbol{q}|^2$, with eigenvector $\boldsymbol{q}$, so this is the expected result.) The complete lift of J (regarded as a (1, 1) tensor) is given by

$$\tilde{J} = q_iq_j\left(\frac{\partial}{\partial q_i}\otimes dq_j + \frac{\partial}{\partial p_j}\otimes dp_i\right) + (p_jq_i - p_iq_j)\frac{\partial}{\partial p_i}\otimes dq_j.$$

A straightforward calculation then gives

$$h^{(2)} = \tfrac{1}{2}(\boldsymbol{q}\cdot\boldsymbol{p})^2 - \tfrac{1}{2}|\boldsymbol{q}|^2|\boldsymbol{p}|^2 - |\boldsymbol{q}^2|V_2$$

and $h^{(m)} = 0$ for $m \geqslant 3$. So, our procedure produces a single further quadratic integral for the system with Hamiltonian h. We recover in this way results obtained by one of us previously by different methods in [13].

Appendix. Some features of complete lifts

If J is a type (1, 1) tensor field on a manifold Q, its complete lift $\tilde{J}$ to T^*Q was defined in [4] by the formula

$$\mathrm{i}_{\tilde{J}(X)}d\theta = \mathrm{i}_X \mathcal{L}_{J^v} d\theta$$

where J^v is the vertical lift (a vector field on T^*Q). (See also [14] for a different definition of the complete lift.) An immediate property is that $\mathrm{i}_{J^v} d\theta = \mathrm{i}_{\tilde{J}}\theta$.

Lemma A.1. *For all $h \in C^\infty(T^*Q)$, we have*

$$\mathrm{i}_{X_h} d_{\tilde{J}}\theta = -d_{\tilde{J}}h.$$

Proof. Making use (consecutively) of the commutator identity $[\mathrm{i}_X, \mathrm{i}_{\tilde{J}}] = \mathrm{i}_{\tilde{J}(X)}$, the definition of $d_{\tilde{J}}$ and the relations just mentioned, we find

$$\begin{aligned} d_{\tilde{J}}h &= \mathrm{i}_{\tilde{J}}dh = -\mathrm{i}_{\tilde{J}}\mathrm{i}_{X_h}d\theta = -\mathrm{i}_{X_h}\mathrm{i}_{\tilde{J}}d\theta + \mathrm{i}_{\tilde{J}(X_h)}d\theta \\ &= -\mathrm{i}_{X_h}d_{\tilde{J}}\theta - \mathrm{i}_{X_h}d\mathrm{i}_{\tilde{J}}\theta + \mathrm{i}_{X_h}\mathcal{L}_{J^v}d\theta \\ &= -\mathrm{i}_{X_h}d_{\tilde{J}}\theta - \mathrm{i}_{X_h}d(\mathrm{i}_{\tilde{J}}\theta - \mathrm{i}_{J^v}d\theta) = -\mathrm{i}_{X_h}d_{\tilde{J}}\theta. \end{aligned}$$

□

Lemma A.2. *$\tilde{J}$ is symmetric with respect to $d\theta$, i.e. for all vector fields X, Y on T^*Q, we have*

$$d\theta(X, \tilde{J}(Y)) = d\theta(\tilde{J}(X), Y).$$

Proof. Using the first-mentioned property, we have

$$\mathrm{i}_{\tilde{J}(X)}d\theta = \mathrm{i}_X d\mathrm{i}_{\tilde{J}}\theta = -\mathrm{i}_X d_{\tilde{J}}\theta + \mathrm{i}_X\mathrm{i}_{\tilde{J}}d\theta.$$

Contracting this with X_h and interchanging the first two contractions in each term, we obtain

$$\mathrm{i}_{\tilde{J}(X)}\mathrm{i}_{X_h}d\theta = -\mathrm{i}_X\mathrm{i}_{X_h}d_{\tilde{J}}\theta + \mathrm{i}_X\mathrm{i}_{X_h}\mathrm{i}_{\tilde{J}}d\theta.$$

Using lemma A.1 and the commutator $[\mathrm{i}_X, \mathrm{i}_{\tilde{J}}]$ in the last term, we obtain the desired property for $Y = X_h$. But since a local frame of vector fields on T^*Q can be constructed out of Hamiltonian vector fields, the result follows for arbitrary Y. □

Recalling now that for a general type (1, 1) tensor field U, $\mathrm{i}_U\omega(X, Y) = \omega(U(X), Y) + \omega(X, U(Y))$, the symmetry of $\tilde{J}$ with respect to ω enables us to eliminate the vector field argument X from the defining relation of $\tilde{J}$, and we obtain $\mathrm{i}_{\tilde{J}}d\theta = 2\mathcal{L}_{J^v}d\theta = 2d\mathrm{i}_{\tilde{J}}\theta$. Hence, we have

$$\omega_1 = d_{\tilde{J}}\theta = d\mathrm{i}_{\tilde{J}}\theta = d\tilde{J}^*\theta$$

which is the same as $d(\hat{J}^*\theta)$, as considered in section 2.

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